INTEGRAL FORMULAE FOR A RIEMANNIAN MANIFOLD WITH A DISTRIBUTION

VLADIMIR ROVENSKI

ABSTRACT. We obtain a new series of integral formulae for symmetric functions of curvature of a distribution of arbitrary codimension (an its orthogonal complement) given on a compact Riemannian manifold, which start from known formula by P. Walczak (1990) and generalize ones for foliations by several authors: Asimov (1978), Brito, Langevin and Rosenberg (1981), Brito and Naveira (2000), Andrzejewski and Walczak (2010), etc. Our integral formulae involve the co-nullity tensor, certain component of the curvature tensor and their products. The formulae also deal with a number of arbitrary functions depending on the scalar invariants of the co-nullity tensor. For foliated manifolds of constant curvature the obtained formulae give us the classical type formulae. For a special choice of functions our formulae reduce to ones with Newton transformations of the co-nullity tensor.

1. INTRODUCTION

1.1. **History.** Let $(M,\langle\cdot,\cdot\rangle)$ be a Riemannian manifold, and D and $D^\perp$ two complementary orthogonal distributions on M (i.e., smooth sub-bundles of the tangent bundle TM). In this paper we assume $n=\dim D$ and $p=\dim D^\perp$. Hence, $\dim M=n+p$. The 2-nd fundamental form B and integrability tensor T of D (and similarly of $D^\perp$) are defined as follows:

$$B(X,Y)=\frac{1}{2}(\nabla_X Y+\nabla_Y X)^\perp,\quad T(X,Y)=\frac{1}{2}(\nabla_X Y-\nabla_Y X)^\perp.$$

D is integrable (i.e., tangent to a foliation) if $T=0$. We call D totally geodesic if $B=0$. Case $B=T=0$ corresponds to a totally geodesic foliation. Denote $H=\operatorname{Tr} B$ and $H^\perp=\operatorname{Tr} B^\perp$ the mean curvature vectors of D and $D^\perp$, resp.

By *integral formula* we mean the vanishing of the integral over M of an expression composed of quantities related to the 2-nd fundamental form and integrability tensor tensors of $D, D^\perp$, and also the curvature tensor of M.

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The first known *integral formula* for codimension one foliations is [**Re**]

$$\int_M H\,d\,\mathrm{vol}=0, \qquad \text{where} \quad H \quad \text{is the mean curvature of leaves.} \tag{1}$$

Its proof is based on the Divergence Theorem, and the identity $\operatorname{div} N=-H$.

Brito, Langevin and Rosenberg [**BLR**] (generalizing pioneer result by Asimov [**A**]) have shown that the integrals of elementary symmetric functions σ_k of principal curvatures a codimension-one foliation $\mathcal{F}$ on a compact M^{n+1} of constant curvature c do not depend on $\mathcal{F}$: they depend on n, k, c and volume of M only,

$$\int_M \sigma_k\,d\,\mathrm{vol}=\begin{cases} c^{k/2}\binom{n/2}{k/2}\mathrm{Vol}(M), & n \text{ and } k \text{ even}\\ 0, & \text{either } n \text{ or } k \text{ odd.}\end{cases} \tag{2}$$

Denote by $K(D,D^\perp)=\sum_{i\le n}\sum_{\alpha\le p} g(R(e_i,e_\alpha)e_i,e_\alpha)$ the *mixed scalar curvature.* A general integral formula for a pair of distributions D and $D^\perp$ on a compact Riemannian manifold M was obtained in [**W**] (for foliation case, see also [**Ra**])

$$\int_M K(D,D^\perp)+\|B\|^2+\|B^\perp\|^2-|H|^2-|H^\perp|^2-\|T\|^2-\|T^\perp\|^2\,d\,\mathrm{vol}=0. \tag{3}$$

The proof is based on the Divergence Theorem, and the identity

$$\operatorname{div}(H+H^\perp)=K(D,D^\perp)+\|B\|^2+\|B^\perp\|^2-|H|^2-|H^\perp|^2-\|T\|^2-\|T^\perp\|^2.$$

For a codimension one foliation with a unit normal N along $D^\perp$, (3) reads as

$$\int_M 2\,\sigma_2-\mathrm{Ric}(N,N)\,d\,\mathrm{vol}=0. \tag{4}$$

The Newton transformations $T_r(N)$ of the shape operator A_N (for N unit) are defined inductively by $T_0(N)=\mathrm{id}$, $T_r(N)=\sigma_r(N)\,\mathrm{id}-A_N\,T_{r-1}(N)$ $(1\le r\le n)$. In [**AW1**], Newton transformations were applied to codimension one foliations, and the formulae, starting from (4), and generalizing series (2), were obtained

$$\int_M \langle \operatorname{div}_{\mathcal{F}} T_r(A_N),\nabla_N N\rangle-(r+2)\sigma_{r+2}+\mathrm{Tr}(T_r(A_N)R_N)\,d\,\mathrm{vol}=0. \tag{5}$$

Here $\langle \operatorname{div}_{\mathcal{F}} T_r(A_N),Z\rangle=\sum_{j=1}^{r}\mathrm{Tr}(R((-A_N)^{j-1}Z)T_{r-j}(A_N))$ for any $Z\in T\mathcal{F}$.

Another integral formulae for codimension one foliations on a Riemannian manifold of finite volume (which are especially nice for a symmetric space) were obtained in [**RW1**], they start from (1) and (4), and also generalize series (2).

Brito and Naveira [**BN**] (see also [**AW2**]) have shown for a totally geodesic foliation (tangent to $D^\perp$) on a compact $M^{n+p}(c)$ that *total extrinsic mean curvatures*, $\gamma_{2s}(D)$, depend on s,p,n,c and the volume of M only. Similar result for *total mean curvatures* $\sigma_{2s}(D)$ (see Definition 1) is obtained in [**RW2**]

$$\sigma_{2s}(D) = \begin{cases} \binom{n/2}{s} \frac{2\pi^{p/2}}{\Gamma(p/2)}\, c^s \operatorname{Vol}(M), & n \text{ even} \\ 0, & n \text{ odd} \end{cases} \tag{6}$$

as a consequence of integral formulae for totally geodesic foliations (of any codimension) on a compact Riemannian space (especially a symmetric space). (Since the distribution D determines a totally geodesic foliation on M, the constant c must be nonnegative.) For a codimension-one distribution D (i.e., $p = 1$), the projection $\pi\colon \S^\perp \to M$ is a double covering. Hence, when D is transversally orientable, using $\Gamma(\frac12) = \sqrt{\pi}$, we reduce (6) to (2) with doubled right hand side.

1.2. **Motivation.** The above integral formulae (1) – (6) are of some interest, they can be useful for the problems: prescribing higher mean curvatures σ_i or other symmetric functions of (the leaves of) a foliation; minimizing functions like volume and energy defined for plane fields on Riemannian manifolds; existence of foliations with all the leaves enjoying a given geometric property such as being totally geodesic, totally umbilical, minimal, etc. (see, among the others, [**RW1**] – [**RW3**], [**W**], [**AW1**], the survey [**Ro**] and the bibliography there). Formula (1) is applied in differential geometry also in the context of harmonic morphisms, holomorphic distributions on Kähler manifolds, see [**S**], and (contact) holomorphicity on almost contact metric manifolds, see [**BS**].

P. Walczak in his lecture "Integral formulae for codim-1 foliations: a final result", given on workshop "Geometry of Foliations" (CRM, Belaterra, 2010) posed the following Question: if (5) can be generalized for a pair of distributions of arbitrary dimensions, as a series of integral formulae depending on Newton transformations related with the distributions, the idea is to compute the divergence of certain vector fields and to apply the Divergence Theorem.

In the work we deduce a series of integral formulae for symmetric functions of curvature of a distribution, which start from (3) of [**W**] and [**Ra**], and generalize and complete ones by [**BLR**], [**BN**], [**AW1**] and [**AW2**]. Our formulae involve the co-nullity tensor, certain components of the curvature tensor and their products. The formulae also include arbitrary functions f_j $(0 \le j < n)$ depending on the scalar invariants of the co-nullity tensor. For a special choice of functions f_j, $f_j = (-1)^j \sigma_{r-j}$, our formulae reduce to ones with Newton transformations of the co-nullity tensor, for codimension-one integrable distribution D the result coincides with (5). For manifolds of constant curvature the obtained formulae are reduced (with a simple choice of f_j) to known formulae (6).

1.3. **Structure.** The work is organized as follows. It starts with Introduction (Section **1**). Section **2** provides some preliminaries (necessary definitions and auxiliary lemmas). Section **3** contains the main results of the work (Propositions 1 and 2 and Theorems 1 and 2 for distributions, foliations and Newton transformations, resp.). Section **4** represents some corollaries and simple examples. The last Section **5** contains applications to codimension-one distributions and foliations, and a particular case of Newtonian transformations. Throughout the work everything (manifolds, distributions, etc.) is assumed to be C^∞-differentiable.

2. Preliminaries

2.1. **Notations.** The *co-nullity (1,2)-tensor* $C\colon D^\perp \times D \to D$ of a distribution D is defined by

$$C_N X = -\nabla_X N^{\top} \qquad (X \in D,\ N \in D^\perp). \tag{7}$$

For a fixed $N \in D^\perp$, C_N is a (1,1)-tensor in D. If D is tangent to a foliation $\mathcal{F}$, then C_N is self-adjoint, moreover, $C_N = 0$ when $\mathcal{F}$ is totally geodesic.

Set $C_N^j = (C_N)^j$ for any integer $j > 1$. Denote by $\tau_j(N) = \operatorname{Tr} C_N^j$ the *power sums* symmetric functions, and set $\vec\tau(N) = (\tau_1(N), \ldots, \tau_n(N))$.

The functions $\tau_j(N)$ can be expressed as polynomials of elementary symmetric functions $\sigma_1, \ldots, \sigma_n$, using the Newton formulae

$$\begin{aligned} &\tau_j - \tau_{j-1}\sigma_1 + \ldots + (-1)^{j-1}\tau_1\sigma_{j-1} + (-1)^j j\,\sigma_j = 0 \quad (1 \le j \le n),\\ &\tau_j - \tau_{j-1}\sigma_1 + \ldots + (-1)^n\,\tau_{j-n}\sigma_n = 0 \quad (j > n), \end{aligned} \tag{8}$$

where $\sigma_0 = 1$ and $\tau_0 = n$. The *elementary symmetric functions* σ_j are given by

$$\sum\nolimits_{j=0}^{n} \sigma_j(N)\, t^j = \det(I_n + t\, C_N).$$

Denote $S^\perp = \{N \in D^\perp \colon \langle N, N\rangle = 1\}$ the unit sphere bundle with the Sasaki metric and the volume form $\omega^\perp$. The natural projection $\pi\colon S^\perp \to M$ is a Riemannian submersion with totally geodesic fibers $\{S^\perp(q)\}_{q\in M}$, which are the unit spheres. For a Riemannian submersion $\pi\colon E \to M$ with totally geodesic fibers F, the volume form is decomposed as $d\operatorname{vol}(E) = d\operatorname{vol}(F)\, d\operatorname{vol}(M)$, see [**B**]. Hence the volume $d\operatorname{vol}(S^\perp)$ is the product of $d\operatorname{vol}(S_1^{p-1})$ and $d\operatorname{vol}(M)$, and the derivation along M commutes with the integration on the fibers $S^\perp(q)$.

Let $f_j \in C^1(\mathbb{R}^n)$ $(0 \le j < n)$ be given functions. They also may depend on a point of M. Define a $(1,1)$-tensor field on D by

$$\mathcal{C} = \int_{N\in S^\perp(q)} \mathcal{C}_N\, d\omega^\perp, \quad \text{where } \mathcal{C}_N = \sum\nolimits_{j=0}^{n-1} f_j(\vec\tau(N))\, C_N^j \quad \text{and} \quad q \in M. \tag{9}$$

The degree of $\mathcal{C}$ is $\deg\mathcal{C} = \max\{j : f_j \not\equiv 0\}$. By the Cayley-Hamilton theorem, one may express C_N^n, using the lower degrees C_N^j with $j < n$.

Remark 1. The powers C_N^j with $j > 1$ in (9) are meaningful: for example,

$$T_i(N) = \sigma_i(N)\operatorname{id} - \sigma_{i-1}(N)\, C_N + \ldots + (-1)^i C_N^i \quad (0 < i < n), \tag{10}$$

the Newton transformation of C_N, depends on all C_N^j $(1 \le j \le i)$.

The choice of the right hand side for $\mathcal{C}_N$ in (9) seems to be natural: the powers C_N^j are the only (1,1)-tensors which can be obtained algebraically from the co-nullity operator C_N, while for integrable D, $\tau_1(N),\dots,\tau_n(N)$ (or, equivalently, $\sigma_1(N),\dots,\sigma_n(N)$) generate all the scalar invariants of C_N.

If D is integrable (i.e., tangent to a foliation $\mathcal{F}$), then B is symmetric, $C_N = A_N$ is the self-adjoint Weingarten operator for N, and (9) represents the symmetric $(1,1)$-tensor field on $T\mathcal{F}$

$$\mathcal{A} = \int_{N\in S^\perp(q)} \mathcal{A}_N\, d\omega^\perp, \quad \text{where } \mathcal{A}_N = \sum\nolimits_{j=0}^{n-1} f_j(\vec\tau(N))A_N^j \ \text{ and } \ q\in M. \tag{11}$$

Certainly, $\langle A_N(X),Y\rangle = \langle B(X,Y),N\rangle$ for $X,Y\in\Gamma(T\mathcal{F})$, and $\tau_j(N) = \operatorname{Tr} A_N^j$.

Notice that the operator $\mathcal{A}_N$ was recently introduced in [**RW0**] for studying *extrinsic geometric* flows on a manifold with a codimension one foliation.

Let e_α $(\alpha\le p)$, e_i $(i\le n)$ be a local orthonormal frame adapted to D and $D^\perp$. If S is a $(1,j)$-tensor field S on M, the divergence $\operatorname{div} S$ is a $(0,j)$-tensor

$$\operatorname{div} S(Y_1,\dots,Y_j) = \sum\nolimits_{i\le n}(\nabla_{e_i}S)(e_i,Y_1,\dots,Y_j) + \sum\nolimits_{\alpha\le p}(\nabla_{e_\alpha}S)(e_\alpha,Y_1,\dots,Y_j)$$

where the derivative of S is a $(1,j+1)$-tensor given by

$$(\nabla_X S)(Y_1,\dots,Y_j) = \nabla_X(S(Y_1,\dots,Y_j)) - \sum\nolimits_{i\le j} S(Y_1,\dots,\nabla_X Y_i,\dots Y_j).$$

Certainly, the partial divergence of S, i.e., along D, is a $(0,j)$-tensor

$$\operatorname{div}_D S(Y_1,\dots,Y_j) = \sum\nolimits_{i\le n}(\nabla_{e_i}S)(e_i,Y_1,\dots,Y_j).$$

Denote by * the conjugation of (1,1)-tensor. Notice that

$$\langle (C_N - C_N^*)X, Y\rangle = \langle C_N X,\, Y\rangle - \langle X,\, C_N Y\rangle = \langle [X,Y],\, N\rangle. \tag{12}$$

For any $X,Y\in TM$, define a linear operator $R_{X,Y}\colon D\to D$ by

$$R_{X,Y}\colon Z \to R(Z,X)Y^\top \qquad (Z\in D), \tag{13}$$

and denote $R_N = R_{N,N}$, i.e., $R_N\colon X\to R(Z,N)N^\top$ for $Z\in D$.

In what follows, the trace will be applied along D, and for short we omit the sign of summing on k. We will also use the identity $\operatorname{div}(fX) = f\operatorname{div} X + X(f)$.

2.2. The divergence of tensors C^{*k} and T_r^*.

Lemma 1. *The divergence of C^{*k} for $k>0$ satisfy the inductive formula*

$$\begin{aligned}(\operatorname{div}_D C^{*k})_N &= C_N^*(\operatorname{div}_D C^{*k-1})_N + \frac{1}{k}\nabla^D\tau_k(N)\\ &\quad -\sum\nolimits_{i\le n} R(N, C_N^{*k-1}e_i)\, e_i{}^\top + \sum\nolimits_{\alpha\le p}(C_{e_\alpha} - C_{e_\alpha}^*)\, C_N^{k-1}\nabla_{e_\alpha}N^\top.\end{aligned} \tag{14}$$

Equivalently, $(\operatorname{div}_D C^{*k})_N$ *for* $k>0$ *is given by the formula*

$$(\operatorname{div}_D C^{*k})_N = \sum\nolimits_{1\le j\le k}\Big[\,\frac{1}{k-j+1}\,C_N^{*j-1}\nabla^D\tau_{k-j+1}(N)$$

$$-\sum\nolimits_{i\le n} C_N^{*j-1}R(N, C_N^{*k-j}e_i)\,e_i^\top + \sum\nolimits_{\alpha\le p} C_N^{*j-1}(C_{e_\alpha} - C^*_{e_\alpha})\,C_N^{k-j}\nabla_{e_\alpha}N^\top\Big]. \tag{15}$$

Moreover, for any vector field $X\in\Gamma(D)$*, we have*

$$\langle(\operatorname{div}_D C^{*k})_N,\,X\rangle = \sum\nolimits_{1\le j\le k}\Big[\frac{1}{k-j+1}\,C_N^{j-1}X(\tau_{k-j+1})(N)$$

$$-\operatorname{Tr}(C_N^{k-j}R_{C_N^{j-1}X,\,N}) + \sum\nolimits_{\alpha\le p}\langle C_N^{*j-1}(C_{e_\alpha} - C^*_{e_\alpha})\,C_N^{k-j}\nabla_{e_\alpha}N^\top,\,X\rangle\Big]. \tag{16}$$

If D *determines a foliation, then (15) and (16) are reduced to*

$$(\operatorname{div}_{\mathcal F}A^k)_N = \sum_{1\le j\le k}\Big[\frac{1}{k-j+1}A_N^{j-1}\nabla^{\mathcal F}\tau_{k-j+1}(N) - \sum_{i\le n}A_N^{j-1}R(N, A_N^{k-j}e_i)e_i^\top\Big],$$

$$\langle(\operatorname{div}_D A^k)_N,\,X\rangle = \sum_{1\le j\le k}\frac{1}{k-j+1}\,A_N^{j-1}X(\tau_{k-j+1})(N) - \operatorname{Tr}(A_N^{k-j}R_{A_N^{j-1}X,\,N}).$$

Proof. The configuration maps T and O were introduced in [**G**] by

$$T_PU = (\nabla_{P^\perp}(U^\perp))^\top + (\nabla_{P^\perp}U^\top)^\perp,\quad O_PU = (\nabla_{P^\top}(U^\top))^\perp + (\nabla_{P^\top}U^\perp)^\top,$$

where $P,U\in TM$. For $Y,Z\in\Gamma(D)$ and $N\in\Gamma(D^\perp)$, we have

$$O_YZ = \nabla_YZ^\perp,\qquad \langle O_YZ, N\rangle = -\langle\nabla_YN, Z\rangle = \langle C_NY, Z\rangle.$$

Similarly, $T_\xi N = \nabla_\xi N^\top$ for $\xi, N\in\Gamma(D^\perp)$ etc.

From the Codazzi equation for distributions, see [**G**],

$$\begin{aligned}\langle R(X,Y)Z,N\rangle &= -\langle(\nabla_XO)_YZ,N\rangle + \langle(\nabla_YO)_XZ,N\rangle\\ &\quad+\langle T(O_XY,Z),N\rangle - \langle T(O_YX,Z),N\rangle\end{aligned}$$

we obtain

$$(\nabla^D_XC)_NY - (\nabla^D_YC)_NX = -R(X,Y)N^\top + \nabla_{[X,Y]^\perp}N^\top. \tag{17}$$

We will verify (14) at a point $q\in M$. One may assume $\nabla_{e_i}N^\perp = 0$ at q. Decomposing $(C^*_N)^k = C^*_N(C^*_N)^{k-1}$ for $k>1$, we get at a point q

$$(\operatorname{div}_D C^{*k})_N = \sum\nolimits_{i\le n}(\nabla^D_{e_i}C^{*k})_N\,e_i = C^*_N(\operatorname{div}_D C^{*k-1})_N$$

$$+\sum\nolimits_{i\le n}(\nabla^D_{e_i}C^*)_NC_N^{*k-1}e_i + \sum\nolimits_{i\le n}(C^*_{\nabla_{e_i}N^\perp}C_N^{*k-1} + C^*_NC^{*k-1}_{\nabla_{e_i}N^\perp} - C^{*k}_{\nabla_{e_i}N^\perp})\,e_i$$

$$= C^*_N(\operatorname{div}_D C^{*k-1})_N + \sum\nolimits_{i\le n}(\nabla^D_{e_i}C^*)_NC_N^{*k-1}e_i. \tag{18}$$

Since the above result (18) is tensorial, it is valid for any point of M. Using (12), (17) and symmetries of the curvature tensor, we compute for $X \in D$,

$$\begin{aligned}
&\sum\nolimits_{i\le n}\langle(\nabla^D_{e_i}C^*)_N(C_N^{*k-1}e_i),\,X\rangle = \sum\nolimits_{i\le n}\langle C_N^{*k-1}e_i,\,(\nabla^D_{e_i}C)_N X\rangle\\
&=\sum\nolimits_{i\le n}\langle C_N^{*k-1}e_i,(\nabla^D_X C)_N\,e_i - R(e_i,X)N + \nabla_{[e_i,X]^\perp}N\rangle = \mathrm{Tr}(C_N^{k-1}(\nabla^D_X C)_N)\\
&-\sum\nolimits_{i\le n}\langle R(N,C_N^{*k-1}e_i)e_i,X\rangle + \sum\nolimits_{\alpha\le p}\langle (C_{e_\alpha}-C^*_{e_\alpha})C_N^{k-1}(\nabla_{e_\alpha}N^\top),\,X\rangle.
\end{aligned}$$

Here we used $[e_i,X]^\perp = \sum_{\alpha\le p}\langle[e_i,X],e_\alpha\rangle e_\alpha = \sum_{\alpha\le p}\langle(C^*_{e_\alpha}-C_{e_\alpha})X,e_i\rangle e_\alpha$ and

$$\langle C_N^{*k-1}e_i,\nabla_{[e_i,X]^\perp}N\rangle = \sum\nolimits_{\alpha\le p}\langle(C_\alpha - C^*_\alpha)C_N^{k-1}\nabla_{e_\alpha}N^\top,\,X\rangle.$$

For $X \in D$, (18) gives us

$$\begin{aligned}
&\langle(\mathrm{div}_D\, C^{*k})_N,\,X\rangle = \langle C^*_N(\mathrm{div}_D\, C^{*k-1})_N,\,X\rangle + \mathrm{Tr}(C_N^{k-1}(\nabla^D_X C)_N)\\
&\quad-\sum\nolimits_{i\le n}\langle R(N,C_N^{*k-1}e_i)e_i,\,X\rangle + \sum\nolimits_{\alpha\le p}\langle(C_{e_\alpha}-C^*_{e_\alpha})C_N^{k-1}(\nabla_{e_\alpha}N^\top),\,X\rangle.
\end{aligned}$$

The above and the identity $\mathrm{Tr}(C_N^{k-1}(\nabla^D_X C)_N) = \frac{1}{k}X(\tau_k)(N)$ for $k>0$ yield (14), see Remark 2 in what follows. By induction, from (14) it follows (15). Finally, we conclude that (16) is a consequence of (15) and

$$\begin{aligned}
\Big\langle \sum\nolimits_{i\le n} C_N^{*j-1}R(N,C_N^{*k-j}e_i)e_i^\top, X\Big\rangle &= \Big\langle\sum\nolimits_{i\le n} C_N^{k-j}R_{C_N^{j-1}X,N}e_i,e_i\Big\rangle\\
&= \mathrm{Tr}(C_N^{k-j}R_{C_N^{j-1}X,N}). \qquad\square
\end{aligned}$$

Remark 2. Let $C(t)$ be a smooth family of n-by-n matrices with the symmetric functions $\tau_j = \mathrm{Tr}\, C^j$. Using the identity $\dot C^k = C\dot C^{k-1} + \dot C C^{k-1}$ for $k>1$, by induction we find $\dot C^k = \sum_{i=1}^k C^{i-1}\dot C C^{k-i}$. By the property $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, and that for matrices the trace commutes with derivative, we conclude that

$$k\,\mathrm{Tr}(C^{k-1}\dot C) = \mathrm{Tr}(\dot C^k) = \dot\tau_k(C). \tag{19}$$

Next lemma was proved in [**AW1**] for $p=1$ and integrable D.

Lemma 2. *Let e_i, e_α be a local orthonormal basis of TM adapted for $D, D^\perp$ such that $\nabla^D_X e_i(q) = 0$ and $\nabla_X e_\alpha(q)^\perp = 0$ for any vector $X \in (TM)_q$. Then for any unit vector $N = \sum_{\alpha\le p} y_\alpha e_\alpha \in S^\perp(q)$ ($y_\alpha \in \mathbb{R}$) we have at the point q*

$$\begin{aligned}
\langle\nabla^D_{e_i}(\nabla_N N),e_j\rangle &= (C_N^2)_{ij} + \langle R(e_i,N)N,e_j\rangle - (\nabla^D_N\, C_N)_{ij}\\
&\quad+\sum\nolimits_{\alpha\le p}\langle\nabla_N e_\alpha,e_i\rangle\langle\nabla_{e_\alpha}N,e_j\rangle.
\end{aligned}\tag{20}$$

Proof. Denote for short $Z = \nabla_N N$. First, observe that

$$-\langle Z,\nabla_{e_i}e_j\rangle = \langle\nabla_{e_i}Z,e_j\rangle + \langle\nabla_{e_i}N,\nabla_N\, e_j\rangle + \langle N,\nabla_{e_i}\nabla_N\, e_j\rangle. \tag{21}$$

We have $(\nabla^D_N C_N)_{ij} = \langle Z, \nabla_{e_i} e_j\rangle + \langle N, \nabla_N \nabla_{e_i} e_j\rangle$. Therefore, we obtain at q

$$(C_N^2)_{ij} + \langle R(e_i, N)N, e_j\rangle - (\nabla^D_N C_N)_{ij} = (C_N^2)_{ij} - \langle R(e_i, N)e_j, N\rangle$$
$$+N(\langle \nabla_{e_i} \nabla_N e_j\rangle) = (C_N^2)_{ij} - \langle Z, \nabla_{e_i} e_j\rangle - \langle \nabla_{e_i} \nabla_N e_j, N\rangle + \langle \nabla_{[e_i,N]} e_j, N\rangle. \tag{22}$$

Using (21), conditions at q,

$$\nabla_{e_i} N = \sum\nolimits_{\alpha\le p} \langle \nabla_{e_i} N, e_k\rangle e_k, \quad \nabla_N e_i = \sum\nolimits_{\alpha\le p} \langle \nabla_N e_i, e_\alpha\rangle e_\alpha,$$

and $(C_N^2)_{ij} = \langle \nabla_{e_i} N, e_k\rangle \langle \nabla_{e_k} e_j, N\rangle$, we simplify the last line in (22) as

$$\langle \nabla_{e_i} Z, e_j\rangle - \sum\nolimits_{\alpha\le p} \langle \nabla_N e_\alpha, e_i\rangle \langle \nabla_{e_\alpha} N, e_j\rangle.$$

From above it follows (20). $\square$

The Newton transformations $T_r(N)$ arising from C_N (for a unit N) may be defined inductively, see also (10), by

$$T_0(N) = \mathrm{id}, \qquad T_r(N) = \sigma_r(N)\,\mathrm{id} - C_N\, T_{r-1}(N) \quad (1 \le r \le n). \tag{23}$$

For example, $T_1(N) = \sigma_1(N)\,\mathrm{id} - C_N$. Notice that C_N and $T_{r-1}(N)$ commute.

Next lemma is known for codimension one foliations (i.e., C_N is symmetric).

Lemma 3. *We have*

$$\begin{aligned} \mathrm{Tr}_D\, T_r(N) &= (n-r)\,\sigma_r(N), \\ \mathrm{Tr}_D(C_N T_r(N)) &= (r+1)\,\sigma_{r+1}(N), \\ \mathrm{Tr}_D(C_N^2 T_r(N)) &= \sigma_1(N)\,\sigma_{r+1}(N) - (r+2)\,\sigma_{r+2}(N), \\ \mathrm{Tr}_D(T_r(N)(\nabla^D_X C)_N) &= X(\sigma_{r+1})(N), \quad X \in TM. \end{aligned}$$

Proof. The first three algebraic properties follow directly from the Newton formulae (8). To prove the last identity, consider a smooth family $C(t)$ of square matrices with the symmetric functions τ_j and σ_j, see Remark 2. Using the derivation formula for elementary symmetric functions, see [**RW0**],

$$\dot\sigma_r = \sum\nolimits_{j=0}^{r-1} \frac{(-1)^j}{j+1}\, \sigma_{r-j-1}\, \dot\tau_{j+1}. \tag{24}$$

one may show that the Newton transformations of C satisfy

$$T_{r-1}(C)\dot C = \sum\nolimits_{j=0}^{r-1} (-1)^j \sigma_{r-j-1} C^j \dot C.$$

Hence, by (19), and (24), we obtain

$$\mathrm{Tr}(T_{r-1}(C)\dot C) = \sum\nolimits_{j=0}^{r-1} (-1)^j \sigma_{r-j-1} \mathrm{Tr}(C^j \dot C) = \sum\nolimits_{j=0}^{r-1} \frac{(-1)^j}{j+1}\, \sigma_{r-j-1} \dot\tau_{j+1} = \dot\sigma_r. \quad \square$$

From Lemma 1 with $f_j = (-1)^j \sigma_{r-j}$ $(j \le r)$, it follows the claim for Newton transformations of C_N, which codimension-one integrable D was obtained in [**AW1**]. For convenience of a reader, we will prove it.

Lemma 4. *The divergence of $T^*_r(N)$ for $r>0$ satisfy the inductive formula*

$$
\begin{aligned}
(\operatorname{div}_D T^*_r)(N) \,=\, & -C^*_N(\operatorname{div}_D T^*_{r-1}(N)) + \sum\nolimits_{i\le n} R(N, T^*_{r-1}(N)e_i)e_i{}^\top \\
& - \sum\nolimits_{e_\alpha\le p}(C_{e_\alpha} - C^*_{e_\alpha})T_{r-1}(N)\nabla_{e_\alpha}N^\top .
\end{aligned} \tag{25}
$$

Equivalently, $\operatorname{div}_D T_r(N)$ *for $r>0$ is given by the formula*

$$
\begin{aligned}
\operatorname{div}_D T^*_r(N) \,=\, & \sum\nolimits_{1\le j\le r}\Big[\sum\nolimits_{i\le n}(-C^*_N)^{j-1}R(N, T^*_{r-j}(N)e_i)e_i^\top \\
& - \sum\nolimits_{\alpha\le p}(-C^*_N)^{j-1}(C_{e_\alpha} - C^*_{e_\alpha})T_{r-j}(N)\nabla_{e_\alpha}N^\top\Big].
\end{aligned} \tag{26}
$$

Moreover, for any vector field $X\in\Gamma(D)$, we have

$$
\begin{aligned}
\langle\operatorname{div}_D T^*_r(N),\, X\rangle \,=\, & \sum\nolimits_{1\le j\le r}\Big[\operatorname{Tr}\big(T_{r-j}R_{(-C_N)^{j-1}X,\,N}\big) \\
& - \sum\nolimits_{\alpha\le p}\langle(-C^*_N)^{j-1}(C_{e_\alpha} - C^*_{e_\alpha})T_{r-j}(N)\nabla_{e_\alpha}N^\top,\, X\rangle\Big].
\end{aligned} \tag{27}
$$

Proof. Using inductive definition (23), we have

$$
(\operatorname{div}_D T^*_r)(N) = \nabla^D\sigma_r(N) - C^*_N(\operatorname{div}_D T^*_{r-1}(N)) - \sum\nolimits_{i\le n}(\nabla^D_{e_i}C^*)_N T^*_{r-1}(N)e_i .
$$

Similarly to the proof of Lemma 1, using Codazzi equation (17), we obtain

$$
\begin{aligned}
& \sum\nolimits_{i\le n}\langle(\nabla^D_{e_i}C^*)_N T^*_{r-1}(N)e_i,\, X\rangle = \sum\nolimits_{i\le n}\langle T^*_{r-1}(N)e_i,\, (\nabla^D_{e_i}C)_N X\rangle \\
& = \sum\nolimits_{i\le n}\big[\langle T^*_{r-1}(N)e_i,\, (\nabla^D_X C)_N e_i - R(e_i, X)N + \nabla_{[e_i,X]^\perp}N\rangle\big] \\
& = \operatorname{Tr}(T_{r-1}(N)(\nabla^D_X C)_N) - \sum\nolimits_{i\le n}\langle R(N, T^*_{r-1}(N)e_i)e_i,\, X\rangle \\
& + \sum\nolimits_{\alpha\le p}\langle(C_{e_\alpha} - C^*_{e_\alpha})T_{r-1}(N)\nabla_{e_\alpha}N^\top,\, X\rangle .
\end{aligned}
$$

By Remark 2, we have $X(\sigma_r)(N) = \operatorname{Tr}(T_{r-1}(N)(\nabla_X C)_N)$ for any $X\in\Gamma(D)$. Hence, the inductive formula (25) holds. Then, (26) directly follows. Finally, from above, for every vector field $X\in\Gamma(D)$, it follows

$$
\begin{aligned}
\langle\operatorname{div}_D T^*_r(N),\, X\rangle \,=\, & \sum\nolimits_{1\le j\le r}\Big[\sum\nolimits_{i\le n}\langle(-C^*_N)^{j-1}R(N, T^*_{r-j}(N)e_i)e_i^\top,\, X\rangle \\
& - \sum\nolimits_{\alpha\le p}\langle(-C^*_N)^{j-1}(C_\alpha - C^*_\alpha)T_{r-j}(N)\nabla_{e_\alpha}N^\top,\, X\rangle\Big].
\end{aligned}
$$

We obtain (27) from above, using the operator (13), and replacing

$$
\sum\nolimits_{i\le n}\langle(-C^*_N)^{j-1}R(N, T^*_{r-j}(N)e_i)e_i^\top,\, X\rangle = \operatorname{Tr}(T_{r-j}R_{(-C_N)^{j-1}X,N}). \qquad \square
$$

3. Main results

In the section we find integral formulae on manifolds with a pair of complementary distributions D and $D^\perp$. Recall that $f_k = f_k(\vec\tau(N))$. The idea is to compute the divergence of a vector field $\int_{S^\perp(q)} \mathcal{C}_N(\nabla_N N)\, d\omega^\perp_q$ (where $q \in M$). For a codimension one distribution D, this is simply $\operatorname{div}(\mathcal{C}_N(\nabla_N N))$.

For a unit vector field N in $S^\perp$, we will denote $Z = \nabla_N N^\top$ for short.

Proposition 1. *Let D be a distribution on M. Then for any $q \in M$,*

$$\operatorname{div}_D \int_{S^\perp(q)} \mathcal{C}_N Z\, d\omega^\perp = \int_{S^\perp(q)} \underline{\langle(\operatorname{div}_D \mathcal{C}^*)_N,\, Z\rangle} + f_k\, \tau_{k+2}(N)$$

$$+\operatorname{Tr}(\mathcal{C}_N R_N) - \frac{f_k}{k+1} N(\tau_{k+1})(N) + \sum\nolimits_{\alpha\le p} \langle \mathcal{C}_N(\nabla_{e_\alpha} N^\top),\, \nabla_N e_\alpha\rangle\, d\omega^\perp, \tag{28}$$

where

$$\langle(\operatorname{div}_D \mathcal{C}^*)_N,\, Z\rangle = C_N^k Z(f_k) + f_k \sum\nolimits_{1\le j\le k} \Big[\frac{1}{k-j+1}\, C_N^{j-1} Z(\tau_{k-j+1})(N)$$

$$-\operatorname{Tr}(C_N^{k-j} R_{C_N^{j-1} Z, N}) + \sum\nolimits_{\alpha\le p} \langle C_N^{*j-1}(C_{e_\alpha} - C^*_{e_\alpha})\, C_N^{k-j}(\nabla_{e_\alpha} N^\top), Z\rangle\Big]. \tag{29}$$

Moreover, if D determines a foliation, then

$$\operatorname{div}_{\mathcal F} \int_{S^\perp(q)} \mathcal{A}_N Z\, d\omega^\perp = \int_{S^\perp(q)} \underline{\langle(\operatorname{div}_{\mathcal F} \mathcal{A})_N,\, Z\rangle} + f_k\, \tau_{k+2}(N)$$

$$+\operatorname{Tr}(\mathcal{A}_N R_N) - \frac{f_k}{k+1} N(\tau_{k+1})(N) + \sum\nolimits_{\alpha\le p} \langle \mathcal{A}_N(\nabla_{e_\alpha} N^\top),\, \nabla_N e_\alpha\rangle\, d\omega^\perp,$$

where

$$\langle(\operatorname{div}_{\mathcal F} \mathcal{A})_N,\, Z\rangle = A_N^k Z(f_k)$$

$$+f_k \sum\nolimits_{1\le j\le k} \Big[\frac{1}{k-j+1} A_N^{j-1} Z(\tau_{k-j+1})(N) - \operatorname{Tr}(A_N^{k-j} R_{A_N^{j-1} Z, N})\Big]. \tag{30}$$

Proof. Assume $\nabla_X N^\perp = 0$ for any $X \in T_q M$ at some point $q \in M$, and compute the divergence of $\mathcal{C}_N Z$,

$$\operatorname{div}_D \int_{S^\perp(q)} \mathcal{C}_N Z\, d\omega^\perp = \sum\nolimits_{i\le n} \int_{S^\perp(q)} \langle \nabla_{e_i}(\mathcal{C}_N Z), e_i\rangle d\omega^\perp$$

$$= \int_{S^\perp(q)} \langle(\operatorname{div}_D \mathcal{C}^*)_N,\, Z\rangle d\omega^\perp + \sum\nolimits_{i\le n} \int_{S^\perp(q)} \langle \nabla_{e_i} Z,\, \mathcal{C}^*_N e_i\rangle d\omega^\perp. \tag{31}$$

The integrand of the first term in (31) is

$$\langle(\operatorname{div}_D \mathcal{C}^*)_N,\, Z\rangle = \langle \nabla^D f_k, C_N^k Z\rangle + f_k \langle(\operatorname{div}_D C^{*k})_N,\, Z\rangle.$$

From above, using (16), we obtain (29).

In order to compute the divergence in (31), we must find $\langle\nabla_{e_i}Z,\mathcal{C}^*_N e_i\rangle$. Using (20) of Lemma 2, we compute the integrand $\langle\nabla_{e_i}Z,\mathcal{C}^*_N e_i\rangle$ in (31), as

$$\begin{aligned}&\langle C^2_N e_i + R(e_i,N)N - (\nabla^D_N C_N)e_i,\, \mathcal{C}^*_N e_i\rangle + \sum\nolimits_{\alpha\le p}\langle\nabla_N e_\alpha, e_i\rangle\langle\nabla_{e_\alpha}N,\mathcal{C}^*_N e_i\rangle\\ &= f_k\tau_{k+2}(N) + \operatorname{Tr}(\mathcal{C}_N R_N) - \operatorname{Tr}(\mathcal{C}_N(\nabla^D_N C)_N) + \sum\nolimits_{\alpha\le p}\langle\mathcal{C}_N(\nabla_{e_\alpha}N^\top),\nabla_N\, e_\alpha\rangle.\end{aligned}$$

We transform $\operatorname{Tr}(\mathcal{C}_N(\nabla^D_N\, C)_N)$, using the definition $\mathcal{C}_N=\sum f_k C^k_N$, as

$$\operatorname{Tr}(\mathcal{C}_N(\nabla^D_N\, C)_N) = \frac{f_k}{k+1}\operatorname{Tr}((\nabla^D_N\, C^{k+1})_N) = \frac{f_k}{k+1}N(\tau_{k+1})(N),$$

see Remark 2. Finally, we obtain

$$\begin{aligned}\langle\nabla_{e_i}Z,\,\mathcal{C}^*_N e_i\rangle &= f_k\tau_{k+2}(N) + \operatorname{Tr}(\mathcal{C}_N R_N) - \frac{f_k}{k+1}N(\tau_{k+1})(N)\\ &\quad+\sum\nolimits_{\alpha\le p}\langle\mathcal{C}_N(\nabla_{e_\alpha}N^\top),\,\nabla_N\, e_\alpha\rangle.\end{aligned}$$

Since the obtained result (28) is tensorial, it is valid for any point q of M. The claim for foliations follows from above directly. $\square$

From Proposition 1 with $f_j=(-1)^j\sigma_{r-j}$ $(j\le r)$, it follows the claim for Newton transformations of C_N, which generalizes result in [**AW1**] for $p=1$ and integrable D, see (5). For convenience of a reader, we will prove it directly.

Proposition 2. *Let D be a distribution on M. Then for any $q\in M$,*

$$\begin{aligned}\operatorname{div}_D\!\int_{S^\perp(q)} T_r(N)Z\,d\omega^\perp &= \int_{S^\perp(q)}\underline{\langle\operatorname{div}_D T^*_r(N),Z\rangle} - N(\sigma_{r+1})(N) - (r+2)\sigma_{r+2}(N)\\ &\quad+\sigma_1(N)\,\sigma_{r+1}(N) + \operatorname{Tr}(T_r(N)\,R_N) + \sum\nolimits_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha}N^\top),\nabla_N\, e_\alpha\rangle d\omega^\perp,\end{aligned}\tag{32}$$

where $Z=\nabla_N N^\top$ (for short) and

$$\begin{aligned}\langle\operatorname{div}_D T^*_r(N),\,Z\rangle &= \sum\nolimits_{1\le j\le r}\big[\operatorname{Tr}\big(T_{r-j}R_{(-C_N)^{j-1}Z,\,N}\big)\\ &\quad-\sum\nolimits_{\alpha\le p}\langle(-C^*_N)^{j-1}(C_{e_\alpha}-C^*_{e_\alpha})T_{r-j}(N)\nabla_{e_\alpha}N^\top,\,Z\rangle\big].\end{aligned}\tag{33}$$

Proof. Notice that (33) is (27) with $X=Z$. Assume $\nabla_X N^\perp=0$ for all $X\in T_qM$ at a point q, and compute the divergence of $T_r(N)Z$,

$$\begin{aligned}\operatorname{div}_D\int_{S^\perp(q)} T_r(N)Z\,d\omega^\perp &= \sum\nolimits_{i\le n}\int_{S^\perp(q)}\langle\nabla_{e_i}(T_r(N)Z),e_i\rangle d\omega^\perp\\ &= \int_{S^\perp(q)}\langle(\operatorname{div}_D T^*_r)(N),\,Z\rangle d\omega^\perp + \sum\nolimits_{i\le n}\int_{S^\perp(q)}\langle\nabla_{e_i}Z,\;T^*_r(N)e_i\rangle d\omega^\perp.\end{aligned}$$

Using (20) of Lemma 2, we compute $\langle\nabla_{e_i}Z,\,T_r^*(N)e_i\rangle$ as

$$\langle C_N^2 e_i + R(e_i,N)N - (\nabla^D_N C_N)e_i,\,T_r^*(N)e_i\rangle + \sum_{\alpha\le p}\langle\nabla_N e_\alpha, e_i\rangle\langle\nabla_{e_\alpha}N, T_r^*(N)e_i\rangle$$
$$= -\operatorname{Tr}\big(T_r(N)(\nabla^D_N C_N - C_N^2 - R_N)\big) + \sum\nolimits_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha}N^\top), \nabla_N\, e_\alpha\rangle.$$

According to Lemma 3, we write down

$$\operatorname{Tr}\big(T_r(N)(\nabla^D_N C_N - C_N^2 - R_N)\big) =$$
$$N(\sigma_{r+1})(N) - \sigma_1(N)\,\sigma_{r+1}(N) + (r+2)\,\sigma_{r+2}(N) - \operatorname{Tr}(T_r(N)\,R_N).$$

Finally, we obtain

$$\begin{aligned}\langle\nabla_{e_i}Z,\,T_r^*(N)e_i\rangle \;&=\; -N(\sigma_{r+1})(N) + \sigma_1(N)\,\sigma_{r+1}(N) - (r+2)\,\sigma_{r+2}(N)\\ &\quad+\;\operatorname{Tr}(T_r(N)\,R_N) + \sum\nolimits_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha}N^\top),\,\nabla_N\,e_\alpha\rangle.\end{aligned}$$

Since the obtained result is tensorial, it is valid for any point of M. This completes the proof of (32), and Proposition 2. $\square$

Notice that for $p>1$ only even values of k are participated in formulae of Proposition 1. On the other hand, using $\sum_{\alpha\le p}(\nabla_{e_\alpha}e_\alpha)^\top = H^\perp$, we have

$$\begin{aligned}\operatorname{div}\int_{S^\perp(q)}\mathcal{C}_N Z\,d\omega^\perp \;&=\; \operatorname{div}_D\Big(\int_{S^\perp(q)}\mathcal{C}_N Z\,d\omega^\perp\Big) + \int_{S^\perp(q)}\sum_{\alpha\le p}\langle\nabla_{e_\alpha}(\mathcal{C}_N Z),\,e_\alpha\rangle\,d\omega^\perp\\ &=\; \operatorname{div}_D\Big(\int_{S^\perp(q)}\mathcal{C}_N Z\,d\omega^\perp\Big) - \int_{S^\perp(q)}\langle\mathcal{C}_N Z,\,H^\perp\rangle\,d\omega^\perp.\end{aligned}\tag{34}$$

Assuming $\nabla_X N^{\,\perp}=0$ for any $X\in T_qM$ at some point $q\in M$, and using the identity $\operatorname{div}N = -\tau_1(N)$, we also find

$$\begin{aligned}\operatorname{div}\Big(\frac{f_k\tau_{k+1}(N)}{k+1}\,N\Big) \;&=\; \frac{f_k}{k+1}\operatorname{div}\big(\tau_{k+1}(N)N\big) + \frac{\tau_{k+1}(N)}{k+1}N(f_k)\\ &=\; \frac{f_k}{k+1}\big[\tau_{k+1}(N)\operatorname{div}N + N(\tau_{k+1})(N)\big] + \frac{\tau_{k+1}(N)}{k+1}N(f_k)\\ &=\; \frac{f_k}{k+1}N(\tau_{k+1})(N) + \frac{\tau_{k+1}(N)}{k+1}\,[N(f_k) - f_k\tau_1(N)].\end{aligned}$$

Thus we have the following theorem

Theorem 1. *Let D be a distribution on a Riemannian manifold M. Then at any point $q\in M$ we have*

$$\begin{aligned}&\operatorname{div}\Big(\int_{S^\perp(q)}\mathcal{C}_N Z\,d\omega^\perp + \frac{f_k\tau_{k+1}(N)}{k+1}N\Big) = \int_{S^\perp(q)}\underline{\langle(\operatorname{div}_D\mathcal{C}^*)_N, Z\rangle} + f_k\tau_{k+2}(N)\\ &+\frac{\tau_{k+1}(N)}{k+1}\,[N(f_k) - f_k\tau_1(N)] + \operatorname{Tr}(\mathcal{C}_N R_N) - \langle\mathcal{C}_N Z, H^\perp\rangle\\ &+\sum\nolimits_{\alpha\le p}\langle\mathcal{C}_N(\nabla_{e_\alpha}N^\top),\,\nabla_N\,e_\alpha\rangle\,d\omega^\perp,\end{aligned}\tag{35}$$

where underlined $\langle \operatorname{div}_D \mathcal{C}^*_N,\, Z\rangle$ *is given by (29). If* M *is compact, then*

$$
\begin{aligned}
&\int_{S^\perp} \underline{\langle (\operatorname{div}_D \mathcal{C}^*)_N,\, Z\rangle} + f_k\,\tau_{k+2}(N) + \frac{\tau_{k+1}(N)}{k+1}\,[N(f_k) - f_k\tau_1(N)]\\
&\quad + \operatorname{Tr}(\mathcal{C}_N R_N) - \langle \mathcal{C}_N Z, H^\perp\rangle + \sum\nolimits_{\alpha\le p} \langle \mathcal{C}_N(\nabla_{e_\alpha} N^\top),\, \nabla_N e_\alpha\rangle\, d\omega^\perp = 0.
\end{aligned}
\tag{36}
$$

Moreover, if D *determines a foliation (on a compact* M*) then*

$$
\begin{aligned}
&\int_{S^\perp} \underline{\langle (\operatorname{div}_{\mathcal F} \mathcal{A})_N,\, Z\rangle} + f_k\,\tau_{k+2}(N) + \frac{f_k}{k+1}[N(f_k) - \tau_1(N)\,\tau_{k+1}(N)]\\
&\quad + \operatorname{Tr}(\mathcal{A}_N R_N) - \langle \mathcal{A}_N Z,\, H^\perp\rangle + \sum\nolimits_{\alpha\le p} \langle \mathcal{A}_N(\nabla_{e_\alpha} N^\top),\, \nabla_N e_\alpha\rangle\, d\omega^\perp,
\end{aligned}
$$

where underlined $\langle (\operatorname{div}_{\mathcal F} \mathcal{A})_N,\, Z\rangle$ *is given by (30).*

The terms $\langle \mathcal{C}_N Z, H^\perp\rangle$ in (35) and (36) have opposite sign and hence cancel.

Remark 3. We will show that integrals over $S^\perp(q)$ when $p>1$ can be reduced to sums. Denote $\boldsymbol{\lambda} = (\lambda_1,\ldots,\lambda_p)$ and $y=(y_1,\ldots,y_p)$. The integrals $I_{\boldsymbol\lambda} := \int_{\|y\|=1} y^{\boldsymbol\lambda}\, d\,\omega_{p-1}$, where $y^{\boldsymbol\lambda} = \prod_{\alpha\le p} y_\alpha^{\lambda_\alpha}$, are given by, see [**PBM**],

$$
I_{\boldsymbol\lambda} = \frac{2}{\Gamma\left(\frac{p}{2} + \frac12\sum_i \lambda_\alpha\right)} \prod\nolimits_{\alpha\le p} \frac12 (1+(-1)^{\lambda_\alpha})\,\Gamma\big(\frac{1+\lambda_\alpha}{2}\big).
$$

Here Γ is the Gamma function. For example,

$$
I_{0,\ldots 0} = \frac{2\,\pi^{p/2}}{\Gamma(p/2)} = \operatorname{Vol}(S^{p-1}_1),\quad I_{2\lambda_1,0,\ldots 0} = 2\,\pi^{\frac{p-1}{2}}\frac{\Gamma(1/2+\lambda_1)}{\Gamma(p/2+\lambda_1)},\ \text{etc.}
$$

For Newton transformations of C_N, similarly to (34), we have

$$
\operatorname{div}\int_{S^\perp(q)} T_r(N) Z\, d\omega^\perp = \operatorname{div}_D\Big(\int_{S^\perp(q)} T_r(N) Z\, d\omega^\perp\Big) - \int_{S^\perp(q)} \langle T_r(N) Z,\, H^\perp\rangle\, d\omega^\perp.
$$

Remark that for all $N\in D^\perp$

$$
\begin{aligned}
\operatorname{div} N &= -\langle H,\, N\rangle = -\tau_1(N),\\
\operatorname{div}(\sigma_{r+1}(N) N) &= -\sigma_1(N)\sigma_{r+1}(N) + N(\sigma_{r+1})(N).
\end{aligned}
$$

Thus, from Proposition 2 (or Theorem 1 for a foliation) we obtain the following theorem (which generalize the result in [**AW1**] for $p=1$ and integrable D).

Theorem 2. *Let* D *be a distribution on a Riemannian manifold* M*. Then at any point* $q\in M$ *we have*

$$
\begin{aligned}
&\operatorname{div}\Big(\int_{S^\perp(q)} (T_r(N) Z + \sigma_{r+1}(N) N)\, d\omega^\perp\Big) = \int_{S^\perp(q)} \underline{\langle (\operatorname{div}_D T^*_r(N), Z\rangle} - (r+2)\sigma_{r+2}(N)\\
&\quad - \langle T_r(N) Z, H^\perp\rangle + \operatorname{Tr}(T_r(N)\, R_N) + \sum\nolimits_{\alpha\le p} \langle T_r(N)(\nabla_{e_\alpha} N^\top),\, \nabla_N e_\alpha\rangle\, d\omega^\perp,
\end{aligned}
$$

where underlined $\langle \operatorname{div}_D T_r^*(N),\, Z\rangle$ *is given by (33). If* M *is compact, then*

$$\begin{aligned}&\int_{S^\perp} \underline{\langle \operatorname{div}_D T_r^*(N), Z\rangle} - \langle T_r(N)Z, H^\perp\rangle - (r+2)\,\sigma_{r+2}(N)\\ &\quad + \operatorname{Tr}(T_r(N)\,R_N) + \sum\nolimits_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha} N^\top), \nabla_N\, e_\alpha\rangle d\omega^\perp = 0.\end{aligned}\tag{37}$$

4. First examples and corollaries

4.1. **Initial members.** First, we will look at initial members of series (36), and, in particular, of (37).

(a) Consider (36) for $\mathcal{C}_N = \mathrm{id}$ ($\deg\mathcal{C} = k = 0$),

$$\int_{S^\perp} \tau_2(N) - \tau_1^2(N) + \operatorname{Tr}(R_N) - \langle Z, H^\perp\rangle + \sum\nolimits_{\alpha\le p}\langle \nabla_{e_\alpha} N^\top, \nabla_N\, e_\alpha\rangle\, d\omega^\perp = 0.\tag{38}$$

Recall that $\tau_1(N) = \langle H, N\rangle$. Let $N = \sum_{\alpha\le p} y_\alpha e_\alpha$ (where $y_\alpha\in\mathbb{R}$) be any unit normal vector field. For a 2-homogeneous on N function $f(N,N) = \sum_{\alpha\beta} f(e_\alpha, e_\beta) y_\alpha y_\beta$ (as is integrand of (38)) we have $\int_{S^\perp(q)} f\, d\omega^\perp = \tilde I_2 \sum_{\alpha\le p} f(e_\alpha, e_\alpha)$, where $\tilde I_2 := I_{2,0,\dots,0} = \int_{S^\perp(q)} y^2\, d\omega^\perp$, see Remark 3. In case of (38),

$$\begin{aligned}&\int_{S^\perp(q)} \tau_2(N) - \tau_1^2(N)\, d\omega^\perp = \tilde I_2\,(\|B^\perp\|^2 - |H|^2),\quad \int_{S^\perp(q)} \langle Z, H^\perp\rangle\, d\omega^\perp = \tilde I_2\,|H^\perp|^2,\\ &\int_{S^\perp(q)} \operatorname{Tr}(R_N)\, d\omega^\perp = \tilde I_2 \sum\nolimits_{\alpha\le p} \operatorname{Tr}(R_{e_\alpha}) = \tilde I_2\, K(D, D^\perp),\\ &\int_{S^\perp(q)} \sum\nolimits_{\alpha\le p} \langle \nabla_{e_\alpha} N^\top,\, \nabla_N\, e_\alpha\rangle\, d\omega^\perp = \tilde I_2\,(\|B^\perp\|^2 - \|T^\perp\|^2).\end{aligned}$$

Hence, 1-st member of (36) for $\mathcal{C}_N = \mathrm{id}$ ($\deg\mathcal{C} = 0$) coincides with (3) of [**W**].

Let $n = 1$. In this case $\langle C_N e_1, e_1\rangle = \langle H, N\rangle = \tau_1(N)$. Certainly, $\mathcal{C}_N = f_0\,\mathrm{id}$ and $\langle(\operatorname{div}_D \mathcal{C})_N, Z\rangle = Z(f_0)$, see (29). Assuming $f_0 = 1$, by Proposition 1, we have along any closed D-curve L,

$$\int_L\int_{S^\perp(q)} \tau_2(N) - N(\tau_1)(N) + K(e_1, N) + \sum\nolimits_{\alpha\le p}\langle \nabla_{e_\alpha} N^\top,\, \nabla_N\, e_\alpha\rangle\, d\omega^\perp d\,\mathrm{vol}_L = 0.$$

(b) Let $\mathcal{C}_N = C_N^2$ ($\deg\mathcal{C} = k = 2$). Then, (36) reads as

$$\begin{aligned}&\int_{S^\perp} \tau_4(N) - \frac{1}{3}\tau_1(N)\,\tau_3(N) + \operatorname{Tr}(C_N^2 R_N) - \langle C_N^2 Z, H^\perp\rangle\\ &\quad + \sum\nolimits_{\alpha\le p}\langle C_N^2(\nabla_{e_\alpha} N^\top),\, \nabla_N\, e_\alpha\rangle + \underline{\langle(\operatorname{div}_D C^{*2})_N,\, Z\rangle}\, d\omega^\perp = 0,\end{aligned}\tag{39}$$

where

$$\langle(\mathrm{div}_D\, C^{*2})_N,\, Z\rangle = \frac{1}{2}\, Z(\tau_2)(N) + C_N Z(\tau_1)(N) - \mathrm{Tr}(C_N R_{Z,N} + R_{C_N Z,N})$$
$$+\sum\nolimits_{\alpha\le p}\langle[(C_{e_\alpha} - C^*_{e_\alpha})\, C_N + C^*_N(C_{e_\alpha} - C^*_{e_\alpha})](\nabla_{e_\alpha} N^\top),\ Z\rangle.$$

The integrand of (39) is a 4-homogeneous function of N. If $D^\perp$ is tangent to a totally geodesic foliation, then $B^\perp = T^\perp = Z = 0$ and (39) reads as

$$\int_{S^\perp} \tau_4(N) - \frac{1}{3}\,\tau_1(N)\,\tau_3(N) + \mathrm{Tr}(C_N^2 R_N)\, d\omega^\perp = 0. \tag{40}$$

As far as (39), so its simple reduction (40), are new results.

(c) From (37) for $r = 0$ we get

$$\int_{S^\perp} -2\,\sigma_2(N) + \mathrm{Tr}(R_N) - \langle Z, H^\perp\rangle + \sum\nolimits_{\alpha\le p}\langle\nabla_{e_\alpha} N^\top, \nabla_N\, e_\alpha\rangle d\omega^\perp = 0. \tag{41}$$

By identity $2\,\sigma_2(N) = \tau_1^2(N) - \tau_2(N)$, (41) is equal to (38), which is (3) of [**W**].

For $r = 2$, (37) gives us a similar to (39) result

$$\int_{S^\perp} -4\,\sigma_4(N) - \langle T_2(N)Z, H^\perp\rangle + \mathrm{Tr}(T_2(N)\, R_N) + \mathrm{Tr}\,\big(T_2(R_{C_N Z,\,N} - R_{Z,\,N})\big)$$
$$-\sum\nolimits_{\alpha\le p}\langle(C_{e_\alpha} - C^*_{e_\alpha})(\sigma_1\,\mathrm{id} - C_N)\nabla_{e_\alpha} N^\top - C^*_N(C_{e_\alpha} - C^*_{e_\alpha})\nabla_{e_\alpha} N^\top,\ Z\rangle$$
$$+\sum\nolimits_{\alpha\le p}\langle T_2(N)(\nabla_{e_\alpha} N^\top), \nabla_N\, e_\alpha\rangle d\omega^\perp = 0. \tag{42}$$

If $D^\perp$ is tangent to a totally geodesic foliation, then (42) reads as

$$\int_{S^\perp} 4\,\sigma_4(N) - \mathrm{Tr}(T_2(N)\, R_N)\, d\omega^\perp = 0. \tag{43}$$

As far as (42), so its simple reduction (43), are new results.

4.2. **Totally geodesic/umbilical foliations.** It is known that a totally geodesic distribution $D^\perp$ (by definition, any geodesic of M that is tangent to $D^\perp$ at one point is tangent to $D^\perp$ at all its points) is characterized by the property: $\nabla_N N^\top = 0$ for all $N \in D^\perp$.

Corollary 1. *Let $D^\perp$ be a totally geodesic distribution on a compact M. Then for any $k \ge 0$ we have (by Theorem 1 with $\mathcal{C} = C^k$ for some $k \ge 0$)*

$$\int_{S^\perp} \tau_{k+2}(N) - \frac{1}{k+1}\,\tau_{k+1}(N)\,\tau_1(N) + \mathrm{Tr}(C_N^k R_N)$$
$$-\sum\nolimits_{\alpha\le p}\langle C_N^k(\nabla_{e_\alpha} N^\top),\ \nabla_{e_\alpha} N^\top\rangle\, d\omega^\perp = 0. \tag{44}$$

For mean curvatures, for any $r \ge 0$ we have (by Theorem 2)

$$\int_{S^\perp} (r+2)\,\sigma_{r+2}(N) - \mathrm{Tr}(T_r(N)\, R_N) - \sum\nolimits_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha} N^\top), \nabla_{e_\alpha} N\rangle\, d\omega^\perp = 0.$$

Moreover, if D is integrable (i.e. determines a totally geodesic), then

$$\int_{S^\perp} \tau_{k+2}(N) - \frac{1}{k+1}\,\tau_{k+1}(N)\,\tau_1(N) + \mathrm{Tr}(C^k_N R_N)\,d\omega^\perp = 0, \tag{45}$$

$$\int_{S^\perp} (r+2)\,\sigma_{r+2}(N) - \mathrm{Tr}(T_r(N)\,R_N)\,d\omega^\perp = 0. \tag{46}$$

Notice that if $\mathcal{C}_N = C^k_N$ for some $k \ge 0$, then the integrand of (44) is a $(k+2)$-homogeneous function of N, and is considered for k even only.

Consider applications of (45).

Corollary 2. *Let $D^\perp$ determines a totally geodesic foliation on a compact M with non-negative definite R_N. If $\tau_{k+1}(N) \equiv 0$ for some even k, (where $N \in D^\perp$ and $\tau_i(N)$ are related to the co-nullity tensor C_N of D), then M locally splits into the product $\mathbb{R}^n \times \mathbb{R}^p$.*

Proof. By (45) (or Theorem 1 with $\mathcal{A}_N = A^k$), we have

$$\int_{S^\perp} \tau_{k+2}(N) + \mathrm{Tr}(A^k_N R_N)\,d\omega^\perp = 0. \tag{47}$$

For k even, one has $\tau_{k+2}(N) \ge 0$ and $A^k_N \ge 0$. Hence, by conditions, $\mathrm{Tr}(A^k_N R_N) \ge 0$, and from (47) it follows $R_N = A_N = 0$ on M. $\square$

Definition 1. The *total k-th mean curvature* $\sigma_k(D)$ and the quantities $\tau_k(D)$ are

$$\sigma_k(D) = \int_{S^\perp} \sigma_k(C_N)\,d\,\omega^\perp, \qquad \tau_k(D) = \int_{S^\perp} \tau_k(C_N)\,d\,\omega^\perp.$$

Notice that always $\sigma_{2s+1}(D) = \tau_{2s+1}(D) = 0$. One may show that just (46) yield (6). Namely, let $D^\perp$ be tangent to a totally geodesic foliation, and the mixed curvature is $c \ge 0$. By (46) and $\mathrm{Tr}\,T_r(N) = (n-r)\,\sigma_r(N)$, we obtain

$$\sigma_{r+2}(D) = \frac{c(n-r)}{r+2}\,\sigma_r(D),$$

where $\sigma_0(D) = \int_{S^\perp} 1\,d\,\omega^\perp = \frac{2\pi^{p/2}}{\Gamma(p/2)}\mathrm{Vol}(M)$. From above, by induction, it follows (6). Remark that for $r > 0$, (46) are different from the result of Theorem 3.2 in [**RW2**] in general, but give the same (6), when the mixed curvature is $c \ge 0$.

The total extrinsic mean curvatures, $\gamma_r(D)$, satisfy the relation, see [**AW2**],

$$\gamma_{r+2}(D) = \frac{c(n-r)(p+r)}{(r+2)(r+1)}\,\gamma_r(D),$$

where $\gamma_0(D) = \sigma_0(D)$. Hence, $\sigma_{2s}(D) = F(s,p)\,\gamma_{2s}(D)$, where $F = \prod_{i=1}^s \frac{p+2i-2}{2i-1}$. Indeed, $\sigma_r(D) = \gamma_r(D)$ for $p = 1$.

Similarly, by (44) with $\tau_1 = 0$, we have $\tau_0(D) = \int_{S^\perp} n\,d\,\omega^\perp = n\frac{2\pi^{p/2}}{\Gamma(p/2)}\mathrm{Vol}(M)$, and $\tau_2(D) = -c\,\tau_0(D)$, etc. Using (44) and Remark 3, by induction we obtain

Corollary 3. *Let D be a minimal distribution (i.e., $H=0$), $D^\perp$ determines a totally geodesic foliation on a compact M, and the mixed curvature $K(X,N)=c=const$ for $X\in D$, $N\in D^\perp$. If n even and $p>1$, then for any $s>0$*

$$\tau_{2s}(D)=\frac{2\pi^{p/2}}{\Gamma(p/2)}\,(-c)^s\,n\,\mathrm{Vol}(M). \tag{48}$$

If $p=1$, n even, and $D^\perp$ is orientable, then for any $s>0$,

$$\tau_{2s}(D)=(-c)^s\,n\,\mathrm{Vol}(M). \tag{49}$$

Notice that for $p=1$, the projection $\pi\colon S^\perp\to M$ is a double covering, and (48) are reduced to (49) with doubled right hand side.

Example 1. Let D determines a *totally umbilical* foliation with the normal curvature of leaves $\lambda(N)=\langle H,N\rangle$ $(N\in S^\perp)$. Then we have $\tau_j(N)=n\lambda^j(N)$ and $A_N=\lambda(N)\,\mathrm{id}$. The Newton transformation is of the form $T_r(N)=\frac{n-r}{n}\,\sigma_r(N)\,\mathrm{id}$.

Suppose that also $D^\perp$ determines a totally umbilical foliation. Then $\nabla_N N=H^\perp$ and $\sum_{\alpha\le p}\langle T_r(N)(\nabla_{e_\alpha}N^\top),\nabla_N e_\alpha\rangle=\langle T_r(N)Z,\,H^\perp\rangle$.

Define the partial Ricci tensor $\mathrm{Ric}_D(X,Y):=\mathrm{Tr}\,R_{X,Y}$ for any $X,Y\in TM$, and assume Einstein type property for some $c\in\mathbb{R}$,

$$\mathrm{Ric}_D(X,Y)=c\langle X,Y\rangle,\qquad X,Y\in TM. \tag{50}$$

Under our assumptions,

$$\mathrm{Ric}_D(N,\nabla_N N)=0,\quad \mathrm{Ric}_D(N,N)=c.$$

Hence $\langle \mathrm{div}_D\,T_r(N),\,Z\rangle=0$, see (33), and $\mathrm{Tr}(T_r(N)\,R_N)=c\frac{n-r}{r}\,\sigma_r(N)$. If M is compact then (37) reads as

$$\int_{S^\perp}(r+2)\,\sigma_{r+2}(N)-c\frac{n-r}{n}\,\sigma_r(N)\,d\omega^\perp=0.$$

Similarly, as in the case of constant mixed curvature, we get the following.
If D and $D^\perp$ determine totally umbilical foliations on a compact M with the property (50) then for $p>1$ (see (6))

$$\sigma_{2s+1}(D)=0,\quad \sigma_{2s}(D)=\begin{cases}\binom{n/2}{s}\frac{2\pi^{p/2}}{\Gamma(p/2)}\left(\frac{c}{n}\right)^s\mathrm{Vol}(M), & n \text{ even}\\ 0, & n \text{ odd}.\end{cases}$$

For $p=1$, (no conditions for $D^\perp$) we have (see (2) and also [**AW2**])

$$\sigma_{2s+1}(D)=0,\quad \sigma_{2s}(D)=\begin{cases}\binom{n/2}{s}\left(\frac{c}{n}\right)^s\mathrm{Vol}(M), & n \text{ even}\\ 0, & n \text{ odd}.\end{cases}$$

5. Codimension one distributions and foliations

Let D be a codimension one transversally orientable distribution with a unit normal N on a Riemannian manifold M. We will briefly discuss applications of main results to this particular case and compare with existing formulae.

For $p=1$ we do not integrate along $S^\perp$. Denote for short $C=C_N$, $\tau_k=\tau_{k+1}(N)$, $A=A_N$, etc. Proposition 1 reads as

Proposition 3. *Let D be a codimension one distribution, and N, $N\perp D$, a unit vector field on a Riemannian manifold M. Then*

$$\operatorname{div}_D\big(\mathcal{C}Z\big)=\underline{\langle\operatorname{div}_D\mathcal{C}^*,\,Z\rangle}+f_k\tau_{k+2}+\operatorname{Tr}(\mathcal{C}R_N)-\frac{f_k}{k+1}N(\tau_{k+1})+\langle\mathcal{C}Z,\,Z\rangle,$$

where

$$\begin{aligned}\langle\operatorname{div}_D\mathcal{C}^*,\,Z\rangle\ &=C^kZ(f_k)+f_k\sum\nolimits_{1\le j\le k}\Big[\frac{1}{k-j+1}\,C^{j-1}Z(\tau_{k-j+1})\\ &-\ \operatorname{Tr}(C^{k-j}R_{C^{j-1}Z,N})+\langle(C-C^*)\,C^{k-j}Z,\,C^{j-1}Z\rangle\Big].\end{aligned}\tag{51}$$

If D is integrable (hence, $C_N=A_N$) then

$$\operatorname{div}_{\mathcal F}(\mathcal{A}Z)=\underline{\langle\operatorname{div}_{\mathcal F}\mathcal{A},Z\rangle}+f_k\,\tau_{k+2}-\frac{f_k}{k+1}N(\tau_{k+1})+\langle\mathcal{A}Z,Z\rangle+\operatorname{Tr}(\mathcal{A}R_N),\tag{52}$$

where

$$\langle\operatorname{div}_{\mathcal F}\mathcal{A},Z\rangle=A^kZ(f_k)+\ f_k\sum_{1\le j\le k}\Big[\frac{1}{k-j+1}\langle A^{j-1}Z(\tau_{k-j+1})-\operatorname{Tr}(A^{k-j}R_{A^{j-1}Z,N})\Big].\tag{53}$$

We have $Z=H^\perp$, and from (34) it follows

$$\operatorname{div}(\mathcal{C}Z)=\operatorname{div}_D(\mathcal{C}Z)-\langle\mathcal{C}Z,\,Z\rangle.$$

For $p=1$, Theorem 1 reads as

Theorem 3. *Let N be a unit vector field on a Riemannian manifold M, $D=N^\perp$. Then we have*

$$\operatorname{div}\Big(\mathcal{C}Z+\frac{f_k\tau_{k+1}}{k+1}N\Big)=\underline{\langle\operatorname{div}_{\mathcal F}\mathcal{C}^*,Z\rangle}+f_k\tau_{k+2}+\operatorname{Tr}(\mathcal{C}R_N)+\frac{\tau_{k+1}}{k+1}(N(f_k)-f_k\tau_1),$$

where underlined $\langle\operatorname{div}_{\mathcal F}\mathcal{C}^*,\,Z\rangle$ is given by (51). If M is compact, then

$$\int_M\underline{\langle\operatorname{div}_{\mathcal F}\mathcal{C}^*,\,Z\rangle}+f_k\,\tau_{k+2}+\frac{\tau_{k+1}}{k+1}\,(N(f_k)-f_k\tau_1)+\operatorname{Tr}(\mathcal{C}R_N)\,d\operatorname{vol}.\tag{54}$$

Moreover, D determines a foliation (hence, $C_N=A_N$), then we have

$$\int_M\underline{\langle\operatorname{div}_{\mathcal F}\mathcal{A},Z\rangle}+f_k\,\tau_{k+2}+\frac{\tau_{k+1}}{k+1}(N(f_k)-f_k\tau_1)+\operatorname{Tr}(\mathcal{A}R_N)\,d\operatorname{vol}=0,\tag{55}$$

where underlined $\langle\operatorname{div}_{\mathcal F}\mathcal{A},Z\rangle$ is given by (53).

Notice that for $f_k = (-1)^k \sigma_{r-k}$, (55) implies (5) of [**AW1**].

From Proposition 3 (integrable case) with $k = 0$ it follows

Corollary 4. *Let* $\mathrm{Ric}(N, N) \geq 0$*. Then along any compact leaf with the property* $N(\tau_1) \leq 0$*, we have* $A = \mathrm{Ric}(N, N) = 0$ *and* $Z = 0$*. Hence, if* $\mathrm{Ric}(N, N) > 0$*, then there are no compact leaves with the property* $N(\tau_1) = 0$*.*

Proof. For $k = 0$, from (52) we obtain

$$\mathrm{div}_{\mathcal{F}} Z = \tau_2 - N(\tau_1) + \mathrm{Ric}(N, N) + \langle Z, Z\rangle \geq 0.$$

Along a compact leaf L, we obtain $A = \mathrm{Ric}(N, N) = Z = 0$. If $\mathrm{Ric}(N, N) > 0$, then the above leads to a contradiction along a compact leaf L,

$$0 < \int_L \tau_2 - N(\tau_1) + \mathrm{Ric}(N, N) + \langle Z, Z\rangle \, d\,\mathrm{vol} = 0. \qquad \square$$

Example 2. We will look at initial members of (54). Recall the identity [**W**]

$$\mathrm{div}(\nabla_N N + \tau_1 N) = \mathrm{Ric}(N, N) + \tau_2 - \tau_1^2. \tag{56}$$

Let $k = 0$. Since $\tau_1^2 - \tau_2 = 2\,\sigma_2$, the integrand of (55) is $-2\,\sigma_2 + \mathrm{Ric}(N, N)$, this yields the formula (4), but now D may be non-integrable.

For $\mathcal{C} = C$, (54) with $k = 1$ reads as

$$\int_M \tau_3 + \mathrm{Tr}(C R_N) - \frac{1}{2}\tau_1\,\tau_2 + Z(\tau_1) - \mathrm{Tr}(R_{Z,N})\, d\,\mathrm{vol} = 0. \tag{57}$$

Using $Z(\tau_1) = \mathrm{div}(\tau_1 Z) - \tau_1 \,\mathrm{div}\, Z$, (56), and the identity $\tau_3 + \frac{1}{2}\tau_1^3 - \frac{3}{2}\tau_1\tau_2 = 3\,\sigma_3$, we rewrite (57) in the form

$$\int_M 3\,\sigma_3 - \tau_1\,\mathrm{Ric}(N, N) + \mathrm{Tr}(C R_N) - \mathrm{Ric}(N, Z)\, d\,\mathrm{vol} = 0 \tag{58}$$

which for integrable D is given in [**AW1**].

Now we return to Newton transformations of C_N when $p = 1$. Denote $C = C_N$, $T_r = T_r(N)$ etc., and recall that $Z = H^\perp$ and $\sigma_1 = \langle H,\, N\rangle$.

From Proposition 2 it follows (for integrable case, $T = 0$, see [**AW1**])

Proposition 4. *Let* D *be a codimension-one distribution on* M*. Then*

$$\mathrm{div}_D(T_r Z) = \langle \mathrm{div}_D\, T_r^*, Z\rangle - N(\sigma_{r+1}) - (r+2)\sigma_{r+2} + \sigma_1\sigma_{r+1} + \mathrm{Tr}(T_r R_N) + \langle T_r Z, Z\rangle,$$

where $Z = \nabla_N N$ *(for short) and*

$$\langle \mathrm{div}_D\, T_r^*,\, Z\rangle = \sum_{1\leq j\leq r}\big[\,\mathrm{Tr}\left(T_{r-j} R_{(-C)^{j-1}Z,\,N}\right) - \langle (-C^*)^{j-1}(C - C^*)T_{r-j}Z,\, Z\rangle\big]. \tag{59}$$

Theorem 2 (or Theorem 3 with a special choice of f_k) for $p = 1$ reads as

Theorem 4. *Let D be a codimension-one distribution on a Riemannian manifold M. Then, denoting $Z = \nabla_N N$, we have*

$$\operatorname{div}(T_r Z + \sigma_{r+1} N) = \underline{\langle \operatorname{div}_D T_r^*, Z\rangle} - (r+2)\,\sigma_{r+2} + \operatorname{Tr}(T_r\, R_N),$$

where underlined $\langle \operatorname{div}_D T_r^*(N),\, Z\rangle$ is given by (59). If M is compact, then

$$\int_M \langle \operatorname{div}_D T_r^*, Z\rangle - (r+2)\,\sigma_{r+2} + \operatorname{Tr}(T_r\, R_N)\, d\operatorname{vol} = 0. \tag{60}$$

Remark 4. For integrable D, Theorem 4 was proved in [**AW1**].
For $r = 0$, (60) coincides with (4), and for $r = 1$, by the skew-symmetry of $C - C^*$, (60) is reduced to (58).

From Theorems 3 and 4 it follows

Corollary 5. *Let N be a unit geodesic vector field on a compact Riemannian manifold M, $D = N^\perp$. Then (by Theorem 3 or Corollary 1)*

$$\int_M \tau_{k+2} - \frac{1}{k+1}\,\tau_1\,\tau_{k+1} + \operatorname{Tr}(C^k R_N)\, d\operatorname{vol} = 0, \qquad k \ge 0.$$

For mean curvatures, we have (by Theorem 4)

$$\int_M (r+2)\,\sigma_{r+2} - \operatorname{Tr}(T_r R_N)\, d\omega^\perp = 0, \qquad r \ge 0.$$

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Vladimir Rovenski
Department of Mathematics
University of Haifa
Mount Carmel, Haifa, 31905
Israel
E-mail address: `rovenski@math.haifa.ac.il`
`http://math.haifa.ac.il/ROVENSKI/rovenski.html`